On polynomials sharing preimages of compact sets, and related questions

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Abstract

In this paper we give a solution of the following problem: under what conditions on infinite compact sets $K_1, K_2 \subset \mathbb{C}$ and polynomials f_1, f_2 the preimages $f_1^{-1}\{K_1\}$ and $f_2^{-1}\{K_2\}$ coincide. Besides, we investigate some related questions. In particular, we show that polynomials sharing an invariant compact set distinct from a point have equal Julia sets.

1 Introduction

Let $f_1(z)$, $f_2(z)$ be complex polynomials and $K_1, K_2 \subset \mathbb{C}$ be finite or infinite compact sets. In this paper we investigate the following problem. Under what conditions on the collection $f_1(z), f_2(z), K_1, K_2$ the preimages $f_1^{-1}\{K_1\}$ and $f_2^{-1}\{K_2\}$ coincide that is

$$f_1^{-1}\{K_1\} = f_2^{-1}\{K_2\} = K \qquad (1)$$

for some compact set $K \subset \mathbb{C}$? Let us mention several particular cases when the answer is known.

The following problem was posed in [22]: whether the equality $f_1^{-1}\{-1, 1\} = f_2^{-1}\{-1, 1\}$ for polynomials of the same degree $f_1(z), f_2(z)$ implies that $f_1(z) = \pm f_2(z)$? This problem was solved in [17], [16]. It was shown that actually for any compact set $K \subset \mathbb{C}$ containing at least 2 points and polynomials of the same degree $f_1(z), f_2(z)$ the equality $f_1^{-1}\{K\} = f_2^{-1}\{K\}$ implies that $f_1(z) = \sigma(f_2(z))$ for some linear function $\sigma(z) = az + b$, $a, b \in \mathbb{C}$, such that $\sigma\{K\} = K$.

For polynomials of arbitrary degrees solutions of the equation $f_1^{-1}\{K\} = f_2^{-1}\{K\}$ for a compact set $K \subset \mathbb{C}$ of the positive logarithmic capacity were described in [6]. Recently this result was extended to an arbitrary infinite compact set K in [7]. It was shown that if K is distinct from a union of circles or a segment and $\deg f_2(z) \geq \deg f_1(z)$ then there exists a polynomial $g(z)$ such that $f_2(z) = g(f_1(z))$ and $g^{-1}\{K\} = K$.

Furthermore, the problem of description of pairs of polynomials $f_1(z), f_2(z)$ sharing the Julia set, studied in [3], [11], [4], [5], [21], [2], [1], also is a particular case of problem (1). Here the answer ([21], [2]) says that whenever the common Julia set J is distinct from a circle or a segment there exists a polynomial $p(z)$

such that J is the Julia set of $p(z)$ and up to a symmetry of J the polynomials $f_1(z)$ and $f_2(z)$ are the iterations of $p(z)$.

Finally, notice that problem (1) absorbs the classical problem of description of commuting polynomials ([12], [10], [19], [8]) since commuting polynomials are known to have equal Julia sets.

In this paper we provide a surprisingly simple description of solutions of equation (1) which in particular permits to treat and reprove all the results mentioned above in the uniform way. Namely, we relate equation (1) to the functional equation

$$g_1(f_1(z)) = g_2(f_2(z)), \qquad (2)$$

where $f_1(z), f_2(z), g_1(z), g_2(z)$ are polynomials. It is easy to see that for any polynomial solution of (2) and any compact set $K_3 \subset \mathbb{C}$ we obtain a solution of (1) setting

$$K_1 = g_1^{-1}\{K_3\}, \quad K_2 = g_2^{-1}\{K_3\}. \qquad (3)$$

In particular, for any "decomposable" polynomial $f_2(z) = g_1(f_1(z))$ and any compact set $S \subset \mathbb{C}$ we have:

$$f_2^{-1}\{S\} = f_1^{-1}\{T\},$$

where $T = g_1^{-1}\{S\}$.

The main result of this paper states that, under a very mild condition on the cardinality of K, all solutions of (1) can be obtained in this way. Moreover, using the Ritt theory of factorisation of polynomials we describe these solutions in a very explicit way.

Theorem 1 *Let $f_1(z)$, $f_2(z)$ be polynomials, $\deg f_1 = d_1$, $\deg f_2 = d_2$, $d_1 \leq d_2$, and $K_1, K_2, K \subset \mathbb{C}$ be compact sets such that (1) holds. Suppose that $\mathrm{card}\{K\} \geq \mathrm{LCM}(d_1, d_2)$. Then, if d_1 divides d_2, there exists a polynomial $g_1(z)$ such that $f_2(z) = g_1(f_1(z))$ and $K_1 = g_1^{-1}\{K_2\}$. On the other hand, if d_1 does not divide d_2, then there exist polynomials $g_1(z), g_2(z)$, $\deg g_1 = d_2/d$, $\deg g_2 = d_1/d$, where $d = \mathrm{GCD}(d_1, d_2)$, and a compact set $K_3 \subset \mathbb{C}$ such that (2),(3) hold. Furthermore, in this case there exist polynomials $\tilde{f}_1(z)$, $\tilde{f}_2(z)$, $W(z)$, $\deg W(z) = d$, such that*

$$f_1(z) = \tilde{f}_1(W(z)), \qquad f_2(z) = \tilde{f}_2(W(z)) \qquad (4)$$

and there exist linear functions $\sigma_1(z)$, $\sigma_2(z)$ such that either

$$\begin{aligned} g_1(z) &= z^c R^{d_1/d}(z) \circ \sigma_1^{-1}, & \tilde{f}_1(z) &= \sigma_1 \circ z^{d_1/d}, & (5) \\ g_2(z) &= z^{d_1/d} \circ \sigma_2^{-1}, & \tilde{f}_2(z) &= \sigma_2 \circ z^c R(z^{d_1/d}), \end{aligned}$$

for some polynomial $R(z)$ and c equal to the remainder after division of d_2/d by d_1/d, or

$$\begin{aligned} g_1(z) &= T_{d_2/d}(z) \circ \sigma_1^{-1}, & \tilde{f}_1(z) &= \sigma_1 \circ T_{d_1/d}(z), & (6) \\ g_2(z) &= T_{d_1/d}(z) \circ \sigma_2^{-1}, & \tilde{f}_2(z) &= \sigma_2 \circ T_{d_2/d}(z), \end{aligned}$$

for the Chebyshev polynomials $T_{d_1/d}(z)$, $T_{d_2/d}(z)$.

As a corollary of theorem 1 we obtain the following simple description of the solutions of (1) with $d_1 = d_2$ (cf. ([16], [17]). In particular, this description implies the results of [11], [4] concerning the polynomials of the same degree sharing the Julia set.

Corollary 1 *If equality (1) holds for polynomials $f_1(z)$, $f_2(z)$ such that $d_1 = d_2$ and at least one of the sets K_1, K_2 contains more than one point then there exists a linear function $\sigma(z)$ such that $f_2(z) = \sigma(f_1(z))$ and $K_2 = \sigma\{K_1\}$.*

As an other corollary of theorem 1 we describe the situations when the preimage of a compact set under a polynomial mapping may have symmetries. This result generalizes the corresponding results of [3], [5] proved under assumption that K is the Julia set of $f(z)$.

Denote by Σ_T the group of linear functions which transform the set $T \subset \mathbb{C}$ to itself.

Corollary 2 *Let $f(z)$ be a polynomial and $K, K_1 \subset \mathbb{C}$ be compact sets such that $K = f^{-1}\{K_1\}$. Then Σ_K is a group of rotations. Furthermore, either K is a union of circles and $f(z) = \sigma_2 \circ z^{d_1} \circ \sigma_1$ for some linear functions $\sigma_1(z)$, $\sigma_2(z)$, or Σ_K is finite and $f(z) = \sigma_2 \circ z^a R(z^b) \circ \sigma_1$ for some linear functions $\sigma_1(z)$, $\sigma_2(z)$ and a polynomial $R(z)$, where b equals the order of Σ_K and $a < b$.*

In the case when $K_1 = K_2$ in (1) the totality of solutions of the corresponding equation

$$f_1^{-1}\{T\} = f_2^{-1}\{T\} = K \tag{7}$$

becomes much smaller in comparison with the general case. Namely, under notation introduced above the following result holds.

Theorem 2 *Let $f_1(z)$, $f_2(z)$ be polynomials such that (7) holds for some infinite compact sets $T, K \subset \mathbb{C}$. Then, if d_1 divides d_2, there exists a polynomial $g_1(z)$ such that $f_2(z) = g_1(f_1(z))$ and $g_1^{-1}\{T\} = T$. On the other hand, if d_1 does not divide d_2, then there exist polynomials $\tilde{f}_1(z)$, $\tilde{f}_2(z)$, $W(z)$, $\deg W(z) = d$, satisfying (4). Furthermore, in this case one of the following conditions holds.*

1) T is a union of circles with the common center and

$$\tilde{f}_1(z) = \sigma \circ z^{d_1/d}, \qquad \tilde{f}_2(z) = \sigma \circ \gamma z^{d_2/d} \tag{8}$$

for some linear function $\sigma(z)$ and $\gamma \in \mathbb{C}$.

2) T is a segment and

$$\tilde{f}_1(z) = \sigma \circ \pm T_{d_1/d}(z), \qquad \tilde{f}_2(z) = \sigma \circ \pm T_{d_2/d}(z), \tag{9}$$

for some linear function $\sigma(z)$ and the Chebyshev polynomials $T_{d_1/d}(z), T_{d_2/d}(z)$.

This result was also obtained in [6], [7]. However, our method is completely different from the method used in these papers. In particular, in our proof we do not use the classification of commuting polynomials that eventually allows us to obtain a new proof of this classification.

Furthermore, we describe the polynomials sharing an invariant compact set that is solutions of the equation

$$f_1^{-1}\{T\} = f_2^{-1}\{T\} = T. \tag{10}$$

where $f_1(z), f_2(z)$ are polynomials and $T \subset \mathbb{C}$ is any compact set.

The theorem 3 below generalizes results of [21], [2] proved under the assumption that T is the Julia set of $f_1(z), f_2(z)$.

Theorem 3 *Let $f_1(z), f_2(z)$ be polynomials and $T \subset \mathbb{C}$ be a compact set such that (10) holds. Then one of the following conditions holds.*

1) T is a union of circles and

$$f_1(z) = \sigma \circ z^{d_1} \circ \sigma^{-1}, \qquad f_2(z) = \sigma \circ \gamma z^{d_2} \circ \sigma^{-1} \tag{11}$$

for some linear function $\sigma(z)$ and $\gamma \in \mathbb{C}$, where $|\gamma| = 1$ whenever T is distinct from a point.

2) T is a segment and

$$f_1(z) = \sigma \circ \pm T_{d_1} \circ \sigma^{-1}, \qquad f_2(z) = \sigma \circ \pm T_{d_2} \circ \sigma^{-1} \tag{12}$$

for some linear function $\sigma(z)$ and the Chebyshev polynomials $T_{d_1}(z), T_{d_2}(z)$.

3) The group Σ_T is finite and there exist a polynomial $p(z)$ and integers s_1, s_2 such that $p^{-1}\{T\} = T$ and

$$f_1(z) = \mu_1 \circ p^{\circ s_1}, \qquad f_2(z) = \mu_2 \circ p^{\circ s_2} \tag{13}$$

for some linear functions $\mu_1(z), \mu_2(z) \in \Sigma_T$.

It was shown in [3], [5] that polynomials sharing the Julia set are closely related to the functional equation

$$f_1(f_2(z)) = \mu(f_2(f_1(z))), \tag{14}$$

where $\mu(z)$ is a linear function. It turns out that the same is true for equation (10). Furthermore, theorem 4 below states that actually polynomials sharing an invariant compact set have the same Julia sets and that any of these properties is equivalent to equation (14) for an appropriate linear function $\mu(z)$. Note that together with theorem 3 this implies in particular the classification of commuting polynomials (cf. [12], [10], [19], [8]).

Theorem 4 *The following conditions are equivalent:*

1) Equality (10) holds for some compact set $T \subset \mathbb{C}$ distinct from a point.

2) Polynomials $f_1(z), f_2(z)$ have the same Julia sets.

3) There exist compact sets $T_1, T_2 \subset \mathbb{C}$ such that $f_1^{-1}\{T_1\} = T_1$, $f_2^{-1}\{T_2\} = T_2$ and equation (14) holds for some $\mu(z) \in \Sigma_{T_1} \cap \Sigma_{T_2}$.

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The approach of this paper is similar to the one introduced by the author for solving the Yang problem cited above. It consists in using a relation between a polynomial $f(z)$ and the n-th polynomial of least deviation $p_n(z)$ on the preimage $f^{-1}\{K\}$ of a compact set $K \subset \mathbb{C}$ (see section 2 below). This relation together with the uniqueness theorem for the n-th polynomial of least deviation and the Ritt theorem permits to reduce equation (1) to equation (2).

The paper is organized as follows. In the second section we recall some classical results about polynomial approximations and prove theorem 2.3 which generalizes the previous results of papers [13], [14], [16], [17]. Although essentially we need only the weaker previous result from [14] we give the proof of theorem 2.3 because we believe that this result is interesting by itself.

In the third section we recall two theorems about polynomial solutions of equation (2) which we use subsequently. In the fourth section using approach described above we give the proofs of theorem 1 and corollaries 1,2.

In the fifth section we prove theorem 2. Here, the idea of the proof is to examine the infinite chain of compact sets and polynomials obtained by repeated use of theorem 1.

Finally, in the sixth section using the obtained results as well as some constructions from the papers on Julia sets cited above we prove theorems 3, 4.

2   Polynomial approximations

Denote by P_n the vector space consisting of polynomials of degrees $\leq n$. It is known (see e.g. [15]) that for any compact set $R \subset \mathbb{C}$ and any complex-valued function $\varphi(z)$ continuous on R there exists a polynomial $p_{n,\varphi}(z) \in P_n$ such that

$$\| \varphi - p_{n,\varphi} \| = \min_{p \in P_n} \| \varphi - p \|, \tag{15}$$

where the symbol $\| g \|$ denotes the uniform norm of the function g on R :

$$\| g \| = \max_{x \in R} |g(z)|.$$

Such a polynomial is called the n-th polynomial of least deviation from φ on R. The n-th polynomial of least deviation from φ is known to be unique whenever R contains at least $n + 1$ points (see e.g. [15]). In case when $\varphi(z) = z^n$ the polynomial $z^n - p_{n-1,\varphi}(z)$ is called the n-th monic polynomial of least deviation from zero on R.

It turns out that for an arbitrary compact set R any polynomial $P(z)$ is the polynomial of least deviation on the set $P^{-1}\{R\}$ whenever R is "centered" at the origin. More precisely, the following theorem holds.

Theorem 2.1 *Suppose that $R \subset \mathbb{C}$ is a compact set such that the disk of the smallest radius which contains R is centered at the origin. Then any monic polynomial $P(z)$ of degree n is the n-th monic polynomial of least deviation from zero on the set $P^{-1}\{R\}$.*

This theorem was proved in [16] where it was applied to the description of solutions of (7) with $d_1 = d_2$.

Note that theorem 2.1 implies the following well known result: the n-th normalized Chebyshev polynomial $T_n(z)$ is the n-th monic polynomial of least deviation from zero on $[-1, 1]$. Indeed, it is enough to observe that the formula $T_n(\cos z) = \cos nz$ implies that $T_n^{-1}[-1, 1] = [-1, 1]$. Similarly, one can deduce that the polynomial z^n is the n-th monic polynomial of least deviation from zero on any union of circles centered at the origin.

A more general than theorem 2.1 result was proved by a different method (actually, earlier) in [14] in connection with the description of polynomials of least deviation on Julia sets.

Theorem 2.2 *Let $R \subset \mathbb{C}$ be a compact set and $T(z)$ be the m-th monic polynomial of least deviation from zero on R. Then for any polynomial $P(z)$ of degree n with leading coefficient c_n the polynomial $T(P(z))/c_n^m$ is the mn-th monic polynomial of least deviation from zero on the set $P^{-1}\{R\}$.*

Finally, some more general result - theorem 2.3 below - was proved in [13]. Nevertheless, the proof was given only under the additional assumption that the so called extremal signature (see e.g. [15]) for $\varphi(z) - p_{m,\varphi}(z)$ on R contains no critical values of $P(z)$. Below, we give the proof in the general case generalizing the method of [16].

Theorem 2.3 *Let $R \subset \mathbb{C}$ be a compact set, $\varphi(z)$ be a continuous function on R, and $p_{m,\varphi}(z)$ be the m-th polynomial of least deviation from $\varphi(z)$ on R. Then for any polynomial $P(z)$ of degree n the polynomial $p_{m,\varphi}(P(z))$ is the $mn+n-1$-th polynomial of least deviation from $\varphi(P(z))$ on the set $P^{-1}\{R\}$.*

Proof of theorem 2.3. For any polynomial $Q(z)$ set

$$Q_P(z) = \frac{1}{n} \sum_{\substack{y \in \mathbb{C}, \\ P(y)=P(z)}} Q(y),$$

where the root y of multiplicity k of $P(y) - P(z) = 0$ is repeated k times.

Clearly,

$$\max_{z \in P^{-1}\{R\}} |\varphi(P(z)) - Q_P(z)| \leq \max_{z \in P^{-1}\{R\}} \sum_{\substack{y \in \mathbb{C}, \\ P(y)=P(z)}} \frac{|\varphi(P(y)) - Q(y)|}{n}.$$

On the other hand, since $P^{-1}\{R\}$ together with a point z contains all the points y such that $P(y) = P(z)$, we have:

$$\max_{z \in P^{-1}\{R\}} \sum_{\substack{y \in \mathbb{C}, \\ P(y)=P(z)}} \frac{|\varphi(P(y)) - Q(y)|}{n} \leq \max_{z \in P^{-1}\{R\}} |\varphi(P(z)) - Q(z)|.$$

Therefore, for any $Q(z)$ the inequality

$$\max_{z \in P^{-1}\{R\}} |\varphi(P(z)) - Q_P(z)| \leq \max_{z \in P^{-1}\{R\}} |\varphi(P(z)) - Q(z)| \qquad (16)$$

holds.

Furthermore, observe that for any polynomial $R(z)$ of degree $< n$ the function $R_P(z)$ is constant. Indeed, for $R(z) = z^j$, $0 \leq j \leq n-1$, this follows from the Newton formulas which express $R_P(z)$ via the symmetric functions S_j, $0 \leq j \leq n-1$, of roots y_i, $1 \leq i \leq n$, of $P(y) - P(z) = 0$ and in general case by the linearity.

Let now $Q(z)$ be a polynomial of an arbitrary degree q and let

$$Q(z) = \sum_{i=0}^{[q/n]} a_i(z) P^i(z)$$

be its P-adic decomposition. Then

$$Q_P(z) = \sum_{i=1}^{[q/n]} a_i P^i(z),$$

where $a_i = a_{i_P}(z)/n$ are constants. Therefore,

$$\max_{z \in P^{-1}\{R\}} |\varphi(P(z)) - \sum Q_P(z)| = \max_{z \in R} |\varphi(z) - \sum_{i=1}^{[q/n]} a_i z^i|. \qquad (17)$$

Suppose now that $q < mn + n$. Then

$$\max_{z \in R} |\varphi(z) - \sum_{i=1}^{[q/n]} a_i z^i| \geq \max_{z \in R} |\varphi(z) - p_{m,\varphi}(z)| = \max_{z \in P^{-1}\{R\}} |\varphi(P(z)) - p_{m,\varphi}(P(z))|. \qquad (18)$$

It follows now from (16),(17) and (18) that

$$\max_{z \in P^{-1}\{R\}} |\varphi(P(z)) - Q(z)| \geq \max_{z \in P^{-1}\{R\}} |\varphi(P(z)) - p_{m,\varphi}(P(z))|.$$

Proof of theorem 2.2. Theorem 2.2 follows from theorem 2.3. Indeed, for $h(z) \in P_m$ we have:

$$\min_{p \in P_m} \| (\varphi + h) - p \| = \min_{p \in P_m} \| (\varphi + h) - (p + h) \| = \min_{p \in P_m} \| \varphi - p \|$$

and $p_{m,\varphi+h}(z) = p_{m,\varphi}(z) + h(z)$. Similarly, for $\beta \in \mathbb{C}$ we have:

$$\min_{p \in P_m} \| \beta\varphi - p \| = \min_{p \in P_m} \| \beta\varphi - \beta p \| = \beta \min_{p \in P_m} \| \varphi - p \|$$

and $p_{m,\beta\varphi}(z) = \beta p_{m,\varphi}(z)$.

Therefore,

$$p_{mn-1,z^{mn}}(z) = \frac{p_{mn-1,c_n^m z^{mn}}(z)}{c_n^m} = \frac{p_{mn-1,P^m + c_n^m z^{mn} - P^m}(z)}{c_n^m} =$$

$$= \frac{p_{mn-1,P^m}(z)}{c_n^m} + z^{mn} - \frac{P^m(z)}{c_n^m} = \frac{p_{m-1,z^m}(P(z))}{c_n^m} + z^{mn} - \frac{P^m(z)}{c_n^m}.$$

Hence,

$$z^{mn} - p_{mn-1,z^{mn}}(z) = \frac{(z^m - p_{m-1,z^m}(z))}{c_n^m} \circ P(z) = \frac{T(P(z))}{c_n^m}.$$

Proof of theorem 2.1. Theorem 2.1 is a particular case of theorem 2.2 since its condition is equivalent to the condition that the first monic polynomial of least deviation from zero on R is z.

3 Solutions of $A(B(z)) = C(D(z))$

In this section we recall two theorems about polynomial solutions of the equation

$$A(B(z)) = C(D(z)) \tag{19}$$

proved in [9], [18] (see also [20], Theorems 5 and 8).

Theorem 3.1 *Let* $A(z), B(z), C(z), D(z)$ *be polynomials such that* (19) *holds. Then there exist polynomials* $V(z), \hat{B}(z), \hat{D}(z)$, *such that*

$$B(z) = \hat{B}(V(z)), \quad D(z) = \hat{D}(V(z)), \quad \deg V(z) = \mathrm{GCD}(\deg B(z), \deg D(z)),$$

and there exist polynomials $U(z), \hat{A}(z), \hat{C}(z)$ *such that*

$$A(z) = U(\hat{A}(z)), \quad C(z) = U(\hat{C}(z)), \quad \deg U(z) = \mathrm{GCD}(\deg A(z), \deg C(z)).$$

The theorem 3.1 reduces the problem of finding the solutions of (19) to the one when $\deg A(z) = \deg D(z)$ and $\deg B(z) = \deg C(z)$ are coprime. The answer to the last question is given by the following "second Ritt theorem".

Theorem 3.2 *Let* $A(z), B(z), C(z), D(z)$ *be non-linear polynomials satisfying* (19) *such that* $a = \deg A(z) = \deg D(z)$ *and* $b = \deg B(z) = \deg C(z)$ *are coprime and* $a > b$. *Then there exist linear functions* $\sigma_1(z), \sigma_2(z), \mu(z), \nu(z)$ *such that either*

$$A(z) = \nu \circ z^c R^b(z) \circ \sigma_1^{-1}, \quad B(z) = \sigma_1 \circ z^b \circ \mu,$$

$$C(z) = \nu \circ z^b \circ \sigma_2^{-1}, \qquad D(z) = \sigma_2 \circ z^c R(z^b) \circ \mu \tag{20}$$

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for some polynomial $R(z)$ and c equal to the remainder after division of a by b, or

$$A(z) = \nu \circ T_a(z) \circ \sigma_1^{-1}, \quad B(z) = \sigma_1 \circ T_b(z) \circ \mu,$$

$$C(z) = \nu \circ T_b(z) \circ \sigma_2^{-1}, \quad D(z) = \sigma_2 \circ T_a(z) \circ \mu \tag{21}$$

for the Chebyshev polynomials $T_a(z), T_b(z)$.

4   Solutions of $f_1^{-1}\{K_1\} = f_2^{-1}\{K_2\} = K$

Proof of theorem 1. Let $p_1(z)$ be the d_2/d-th monic polynomial of least deviation from zero on K_1 and $p_2(z)$ be the d_1/d-th monic polynomial of least deviation from zero on K_2. Then by theorem 2.2 the polynomial $p_1(f_1(z))/a_1^{d_2/d}$, where a_1 is the leading coefficient of $f_1(z)$, is the $d_1 d_2/d$-th monic polynomial of least deviation from zero on K. Similarly, the polynomial $p_2(f_2(z))/a_2^{d_1/d}$, where a_2 is the leading coefficient of $f_1(z)$, is the $d_1 d_2/d$-th monic polynomial of least deviation from zero on K. Since

$$\mathrm{card}\{K\} \geq \mathrm{LCM}(d_1, d_2) = d_1 d_2/d$$

it follows from the uniqueness of the polynomial of least deviation that

$$\hat{g}_1(f_1(z)) = \hat{g}_2(f_2(z)), \tag{22}$$

where $\hat{g}_1(z) = p_1(z)/a_1^{d_2/d}$, $\hat{g}_2(z) = p_2(z)/a_2^{d_1/d}$. Hence, by theorem 3.1 there exist polynomials $\tilde{f}_1(z), \tilde{f}_2(z), V(z)$ such that

$$f_1(z) = \tilde{f}_1(V(z)), \quad f_2(z) = \tilde{f}_2(V(z)),$$

where $\deg V(z) = d$.

If d_1 divides d_2 then the polynomial $\tilde{f}_1(z)$ is linear and setting $g_1(z) = \tilde{f}_2 \circ \tilde{f}_1^{-1}$ we see that $f_2(z) = g_1(f_1(z))$. Moreover, since for any polynomial $f(z)$ and sets $T_1, T_2 \subset \mathbb{C}$ the equality $f^{-1}\{T_1\} = f^{-1}\{T_2\}$ implies that $T_1 = T_2$, it follows from the equality

$$f_2^{-1}\{K_2\} = f_1^{-1}\{g_1^{-1}\{K_2\}\} = f_1^{-1}\{K_1\} = K$$

that $K_1 = g_1^{-1}\{K_2\}$.

Furthermore, if d_1 does not divide d_2 then both $\tilde{f}_1(z), \tilde{f}_2(z)$ are non-linear and therefore $\hat{g}_1(z), \hat{g}_2(z)$ are also non-linear. Since equality (22) implies the equality

$$\hat{g}_1(\tilde{f}_1(z)) = \hat{g}_2(\tilde{f}_2(z)), \tag{23}$$

where $\deg \hat{g}_1(z) = \deg \tilde{f}_2(z)$ and $\deg \hat{g}_2(z) = \deg \tilde{f}_1(z)$ are coprime, applying theorem 3.2 to (23) and setting

$$g_1(z) = \nu^{-1} \circ \hat{g}_1(z), \quad g_2(z) = \nu^{-1} \circ \hat{g}_2(z), \quad W(z) = \mu \circ V$$

we see that (2) and (4) hold with $\tilde{f}_1(z), \tilde{f}_2(z), g_1(z), g_2(z)$ satisfying either (5) or (6).

Observe now that

$$g_1\{K_1\} = g_1\{f_1\{K\}\} = g_2\{f_2\{K\}\} = g_2\{K_2\}.$$

Set $K_3 = g_1\{K_1\} = g_2\{K_2\}$ and show that the equalities

$$g_1^{-1}\{K_3\} = K_1, \quad g_2^{-1}\{K_3\} = K_2$$

hold. Notice that it is enough to prove only one of these equalities. Indeed, (2) implies that

$$f_1^{-1}\{g_1^{-1}\{K_3\}\} = f_2^{-1}\{g_2^{-1}\{K_3\}\}. \tag{24}$$

Therefore, if say $g_1^{-1}\{K_3\} = K_1$ then (1) and (24) imply that

$$K = f_2^{-1}\{g_2^{-1}\{K_3\}\}.$$

Since $K = f_2^{-1}\{K_2\}$ it follows that $g_2^{-1}\{K_3\} = K_2$.

Show first that if (5) holds then

$$g_2^{-1}\{K_3\} = K_2. \tag{25}$$

Clearly, equality (25) is equivalent to the equality

$$\sigma_2^{-1}\{K_2\} = (z^{d_1/d})^{-1}\{K_3\}.$$

On the other hand, the last equality is equivalent to the statement that the set $\sigma_2^{-1}\{K_2\}$ together with a point x contains any point of the form εx, where ε is a d_1/d-th root of unity.

In order to prove the last statement first observe that in view of (1) and (4) we have:

$$W^{-1}\{\tilde{f}_1^{-1}\{K_1\}\} = W^{-1}\{\tilde{f}_2^{-1}\{K_2\}\}.$$

Hence

$$\tilde{f}_1^{-1}\{K_1\} = \tilde{f}_2^{-1}\{K_2\} = W\{K\}$$

or equivalently

$$(z^{d_1/d})^{-1}\{\sigma_1^{-1}\{K_1\}\} = (z^c R(z^{d_1/d}))^{-1}\{\sigma_2^{-1}\{K_2\}\} = W\{K\}. \tag{26}$$

Suppose now that $x \in \sigma_2^{-1}\{K_2\}$ and let y be a point of $W\{K\}$ such that $y^c R(y^{d_1/d}) = x$. Then equality (26) implies that any point of the form εy, where ε is a d_1/d-th root of unity, also belongs to $W\{K\}$. Since

$$\sigma_2^{-1}\{K_2\} = z^c R(z^{d_1/d})\{W\{K\}\}$$

it follows that $\sigma_2^{-1}\{K_2\}$ together with a point x contains any point of the form $\varepsilon^c y^c R(y^{d_1/d}) = \varepsilon^c x$. To finish the proof it is enough to observe that the equality $\mathrm{GCD}(d_1/d, d_2/d) = 1$ implies the equality $\mathrm{GCD}(c, d_1/d) = 1$. Therefore, if ε runs all d_1/d-th roots of unity then ε^c also runs all d_1/d-th roots of unity.

In the case when (6) holds the proof of the equality

$$\sigma_2^{-1}\{K_2\} = (T_{d_1/d})^{-1}\{K_3\}$$

which in this case is equivalent to equality (25) is similar. We must show that for any point $x \in \sigma_2^{-1}\{K_2\}$ all the points y such that $T_{d_1/d}(y) = T_{d_1/d}(x)$ also belong to $\sigma_2^{-1}\{K_2\}$. Equivalently, we must show that if $\cos\alpha = x \in \sigma_2^{-1}\{K_2\}$ for some $\alpha \in \mathbb{C}$ then for any $k = 1, 2, ..., (d_1/d) - 1$ the number $\cos\left(\alpha + \frac{2\pi d}{d_1}k\right)$ also belongs to $\sigma_2^{-1}\{K_2\}$.

As above observe that

$$(T_{d_1/d})^{-1}\{\sigma_1^{-1}\{K_1\}\} = (T_{d_2/d})^{-1}\{\sigma_2^{-1}\{K_2\}\} = W\{K\}. \tag{27}$$

Suppose now that $\cos\alpha = x \in \sigma_2^{-1}\{K_2\}$ and set $t = \cos\left(\frac{\alpha d}{d_2}\right)$. Then $T_{d_2/d}(t) = x$ and hence $t \in W\{K\}$. Therefore, (27) implies that all the points of the form

$$\cos\left(\frac{\alpha d}{d_2} + \frac{2\pi d}{d_1}j\right), \quad j = 1, 2, ..., \frac{d_1}{d} - 1,$$

belong to $W\{K\}$. It follows now from the equality

$$\sigma_2^{-1}\{K_2\} = T_{d_2/d}\{W\{K\}\}$$

that all the points of the form

$$\cos\left(\alpha + \frac{2\pi d_2}{d_1}j\right), \quad j = 1, 2, ..., \frac{d_1}{d} - 1,$$

belong to $\sigma_2^{-1}\{K_2\}$. Since the numbers d_2/d and d_1/d are coprime this implies that for any $k = 1, 2, ..., (d_1/d) - 1$ the number $\cos\left(\alpha + \frac{2\pi d}{d_1}k\right)$ belongs to $\sigma_2^{-1}\{K_2\}$.

Remark. Instead of the condition

$$\operatorname{card}\{K\} \geq \operatorname{LCM}\{d_1, d_2\} \tag{28}$$

in the formulation of the theorem one can require that

$$\operatorname{card}\{K_1\} \geq d_2/d + 1 \quad \text{or} \quad \operatorname{card}\{K_2\} \geq d_1/d + 1. \tag{29}$$

Indeed, for any polynomial $f(z)$ and any finite set $K \subset \mathbb{C}$ we have:

$$\operatorname{card}\{f^{-1}\{K\}\} \geq \deg f(z)\operatorname{card}\{K\} - \deg f'(z) =$$

$$= \deg f(z)(\operatorname{card}\{K\} - 1) + 1. \tag{30}$$

Therefore, any of inequalities (29) implies inequality (28).

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Proof of corollary 1. It is enough to observe that if $\operatorname{card}\{K_1\} \geq 2$ then (30) implies that

$$\operatorname{card}\{K_1\} \geq \deg f_1(z) + 1 = \operatorname{LCM}(d_1, d_2) + 1.$$

Proof of corollary 2. Let C be the circle of the smallest radius containing the set K and c be its center. Observe that any $\mu(z) \in \Sigma_K$ transforms C to itself. Therefore, Σ_K is a subgroup of the group S^1. Since K is a compact set, it follows that if Σ_K is infinite then K contains with a point x all the circle with the center c containing x and hence is a union of circles. Moreover, setting

$$\tilde{f}(z) = z^{d_1} \circ \sigma_1, \quad \tilde{K}_1 = \tilde{f}\{K\}$$

where $\sigma_1(z) = z - c$ we see that then

$$\tilde{f}^{-1}\{\tilde{K}_1\} = f^{-1}\{K_1\} = K.$$

It follows now from corollary 1 that $f(z) = \sigma_2 \circ \tilde{f}(z)$ for some linear function $\sigma_2(z)$ and hence $f(z) = \sigma_2 \circ z^{d_1} \circ \sigma_1$.

Suppose now that the group Σ_K is finite. Without loss of generality we can suppose that $c = 0$. Then Σ_K is generated by $\varepsilon_b = exp(2\pi i/b)$. Since

$$f^{-1}\{K_1\} = (f \circ \varepsilon_b z)^{-1}\{K_1\} = K,$$

it follows from corollary 1 that

$$f(\varepsilon_b z) = \mu \circ f(z) \tag{31}$$

for some $\mu(z) \in \Sigma_{K_1}$. If $\mu(z) = \alpha z + \beta$, $\alpha, \beta \in \mathbb{C}$, then $\alpha = \varepsilon_b^a$, where a is the remainder after division of $\deg f(z)$ by b. This implies in particular the equality $f(0) = (f(0) - \beta)/\varepsilon_b^a$. Consider now the rational function $g(z) = (f(z) - f(0))/z^a$. Since

$$g(\varepsilon_b z) = \frac{f(\varepsilon_b z) - f(0)}{\varepsilon_b^a z^a} = \frac{f(z) - (f(0) - \beta)/\varepsilon_b^a}{z^a} = g(z)$$

it is easy to see that $g(z)$ has the form $R(z^b)$ for some polynomial $R(z)$ and hence $f(z) = \sigma \circ z^a R(z^b)$, where $\sigma(z) = z + f(0)$.

5 Solutions of $f_1^{-1}\{T\} = f_2^{-1}\{T\} = K$

Proof of theorem 2. If d_1 is a divisor of d_2 then the theorem follows from theorem 1 so we may concentrate on the case when d_1 is not a divisor of d_2. Let us suppose additionally that $d_1/d > 2$; the case when $d_1/d = 2$ will be considered separately.

Since d_1 is not a divisor of d_2 theorem 1 implies that there exist non-linear polynomials $\tilde{f}_1(z)$, $\tilde{f}_2(z)$, $g_1(z)$, $g_2(z)$, $\deg g_1(z) = d_2/d$, $\deg g_2(z) = d_1/d$, and a polynomial $W(z)$, $\deg W(z) = d$, satisfying (2), (4). Since in course of the

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proof of the theorem we will repeatedly use theorem 1 to uniform the notation set

$$L_1 = T, \quad A_1(z) = \tilde{f}_1(z), \quad B_1(z) = \tilde{f}_2(z).$$

It follows from (2), (4) that

$$A_1^{-1}\{L_1\} = B_1^{-1}\{L_1\} = W\{K\}.$$

Furthermore, by theorem 1 there exists a compact subset of $\mathbb{C}$, which we denote by L_2, such that

$$g_1^{-1}\{L_2\} = g_2^{-1}\{L_2\} = L_1 \tag{32}$$

and there exist linear functions $\sigma_1(z), \sigma_2(z)$ such that either

$$A_1(z) = \sigma_1 \circ z^{d_1/d}, \quad B_1(z) = \sigma_2 \circ z^c R(z^{d_1/d}), \tag{33}$$

or

$$A_1(z) = \sigma_1 \circ T_{d_1/d}, \quad B_1(z) = \sigma_2 \circ T_{d_2/d} \tag{34}$$

holds.

Set now

$$A_2(z) = g_2(\tilde{f}_1(z)), \qquad B_2(z) = g_1(\tilde{f}_2(z)). \tag{35}$$

Then the conditions of the theorem imply that

$$A_2^{-1}\{L_2\} = B_2^{-1}\{L_2\} = W\{K\}, \tag{36}$$

where $\deg A_1(z) = (d_1/d)^2$, $\deg B_2(z) = (d_2/d)^2$. Furthermore, applying theorem 1 to equality (32) we conclude that there exist polynomials $h_1(z), h_2(z)$, $\deg h_1(z) = d_1/d$, $\deg h_2(z) = d_2/d$. such that

$$h_1(g_1(z)) = h_2(g_2(z))$$

and

$$h_1^{-1}\{L_3\} = h_2^{-1}\{L_3\} = L_2$$

for some compact set $L_3 \subset \mathbb{C}$. Then for polynomials

$$A_3(z) = h_1 \circ g_2 \circ f_1, \quad B_3(z) = h_2 \circ g_1 \circ f_2$$

we have:

$$A_3^{-1}\{L_3\} = B_3^{-1}\{L_3\} = W\{K\},$$

where $\deg A_3(z) = (d_1/d)^3$, $\deg B_3(z) = (d_2/d)^3$.

Continuing in the same way we conclude that for any $r \geq 0$ there exist a compact set L_r and polynomials $A_r(z), B_r(z)$, $\deg A_r(z) = (d_1/d)^r$, $\deg B_2(z) = (d_2/d)^r$, such that

$$A_r^{-1}\{L_r\} = B_r^{-1}\{L_r\} = W\{K\}, \tag{37}$$

where for polynomials $A_1(z), B_1(z)$ either (33) or (34) holds. Furthermore, applying theorem 1 to equality (37) for $r \geq 2$ we see that there exist linear functions $\sigma_{r,1}(z)$, $\sigma_{r,2}(z)$, $\omega_r(z)$ such that either

$$A_r(z) = \sigma_{r,1} \circ z^{(d_1/d)^r} \circ \omega_r, \quad B_r(z) = \sigma_{r,2} \circ z^{c_r} R_r(z^{(d_1/d)^r}) \circ \omega_r \qquad (38)$$

for some $R_r(z)$ and c_r, or

$$A_r(z) = \sigma_{r,1} \circ T_{(d_1/d)^r} \circ \omega_r, \quad B_r(z) = \sigma_{r,2} \circ T_{(d_2/d)^r} \circ \omega_r. \qquad (39)$$

Show that if (33) (resp. (34)) holds then (38) (resp. (39)) holds for all $r \geq 2$. Consider first the case when (34) holds and show that equality (38) can not be realized. Indeed, observe that the formula $T_n(\cos x) = \cos(n x)$ implies that $T_n'(z) = 0$ if and only if $z = \cos(\pi k/n)$, where $k = 1, 2, ..., n - 1$. In particular, since $d_1/d > 2$, the polynomial $T_{d_1/d}(z)$ has at least two critical points. It follows now from the chain rule that the polynomial $A_r(z)$, which is by construction a polynomial in $T_{d_1/d}(z)$, also has at least two critical points. On the other hand, the polynomial $\sigma_{1,r} \circ z^{(d_1/d)^r} \circ \omega_r$ has only one critical point.

Similarly, if formula (33) holds then (39) can not be realized since $A_r(z)$ is a polynomial in $z^{d_1/d}$ and therefore has at least one critical point of the multiplicity > 2 while the multiplicity of any critical point of the polynomial $\sigma_{1,r} \circ T_{(d_1/d)^r} \circ \omega_r$ is 2.

Consider now the cases when (38) or (39) holds separately. Suppose first that (39) holds. Show at the beginning that for any $r \geq 2$ the equality

$$\omega_r(z) = \pm z \qquad (40)$$

holds. Indeed, we have:

$$A_r(z) = (\sigma_{r,1} \circ T_{(d_1/d)^{r-1}}) \circ (T_{d_1/d} \circ \omega_r).$$

Therefore, setting

$$U_r = (\sigma_{r,1} \circ T_{(d_1/d)^{r-1}})^{-1}\{L_r\},$$

we see that

$$(T_{d_1/d} \circ \omega_r)^{-1}\{U_r\} = (\sigma_1 \circ T_{d_1/d})^{-1}\{L_1\}.$$

By corollary 1, this implies that

$$T_{d_1/d} \circ \omega_r = \delta \circ \sigma_1 \circ T_{d_1/d}$$

for some linear function $\delta(z)$. Since both parts of this equality should have the same critical points it follows easily that (40) holds.

Furthermore, since $T_n(\pm z) = \pm T_n(z)$ equality (40) implies that

$$A_r(z) = \tilde{\sigma}_{r,1} \circ T_{(d_1/d)^r}, \quad B_r(z) = \tilde{\sigma}_{r,2} \circ T_{(d_2/d)^r},$$

for linear functions $\tilde{\sigma}_{r,1} = \pm \sigma_{r,1}$, $\tilde{\sigma}_{r,2} = \pm \sigma_{r,2}$. In particular, setting $M_1 = \sigma_1^{-1}\{L_1\}$ and $M_r = \tilde{\sigma}_{r,1}^{-1}\{L_r\}$ for $r \geq 2$, we see that for any $r \geq 1$ the equality

$$(T_{(d_1/d)^r})^{-1}\{M_r\} = W\{K\} \qquad (41)$$

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holds.

The equality (41) implies that the compact set $W\{K\}$ together with a point u contains all the points v such that

$$T_{(d_1/d)^r}(v) = T_{(d_1/d)^r}(u) \tag{42}$$

for some $r \geq 1$. Choose $\alpha \in \mathbb{C}$ such that $u = \cos \alpha$. Then condition (42) is equivalent to the condition that $W\{K\}$ contains all the points of the form

$$\cos\left(\alpha + 2\pi \left(\frac{d}{d_1}\right)^r j\right), \quad j = 1, 2, ..., \left(\frac{d_1}{d}\right)^r - 1,$$

where $r \geq 1$. Since $W\{K\}$ is a compact set it follows that $W\{K\}$ contains all the set $E_\alpha = \cos(\alpha + s)$, $0 \leq s \leq 2\pi$. It is easy to see that E_α is an ellipse which in the coordinates $x = \Re z, y = \Im z$ is defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a = \frac{1}{2}\left(|e^{i\alpha}| + \frac{1}{|e^{i\alpha}|}\right), \quad b = \frac{1}{2}\left(|e^{i\alpha}| - \frac{1}{|e^{i\alpha}|}\right).$$

Therefore, we can represent $W\{K\}$ as a union of ellipses

$$W\{K\} = \bigcup_{t \in U} E_t$$

for some compact subset U of the segment $[0, i\infty)$. Furthermore, since $T_n\{E_t\} = E_{tn}$ we have:

$$T = \sigma_1\{T_{d_1/d}\{W\{K\}\}\} = \bigcup_{t \in U} \sigma_1\{E_{td_1/d}\}. \tag{43}$$

On the other hand,

$$T = \sigma_2\{T_{d_2/d}\{W\{K\}\}\} = \bigcup_{t \in U} \sigma_2\{E_{td_2/d}\}. \tag{44}$$

Denote by t_1 the point of U with the maximal modulus. Then formulas (43), (44) imply that the ellipses $\sigma_1\{E_{t_1 d_1/d}\}$ and $\sigma_2\{E_{t_1 d_2/d}\}$ coincide. In particular they have the same focuses. Since focuses of all ellipses $E_\alpha, \alpha \in \mathbb{C}$ are ± 1 we conclude that $\sigma_2 \circ \sigma_1^{-1} = \pm z$ and hence (9) holds. Furthermore, the equality

$$\sigma_1\{E_{t_1 d_1/d}\} = \sigma_1\{\pm E_{t_1 d_2/d}\}$$

implies that

$$\frac{1}{2}\left(|e^{it_1}|^{d_1/d} + \frac{1}{|e^{it_1}|^{d_1/d}}\right) = \frac{1}{2}\left(|e^{it_1}|^{d_2/d} + \frac{1}{|e^{it_1}|^{d_2/d}}\right).$$

Since $d_2 \neq d_1$ this follows $t_1 = 0$. Therefore, $W\{K\} = [-1, 1]$ and hence

$$T = \sigma_1\{T_{d_1/d}\{[-1, 1]\}\} = \sigma_1\{[-1, 1]\}$$

is a segment.

Consider now the case when (38) holds. Since

$$A_r(z) = (\sigma_{r,1} \circ z^{(d_1/d)^{r-1}}) \circ (z^{d_1/d} \circ \omega_r),$$

setting

$$U_r = (\sigma_{r,1} \circ z^{(d_1/d)^{r-1}})^{-1}\{L_r\},$$

we see that

$$(z^{d_1/d} \circ \omega_r)^{-1}\{U_r\} = (\sigma_1 \circ z^{d_1/d})^{-1}\{L_1\}.$$

By corollary 1, this implies that

$$z^{d_1/d} \circ \omega_r = \delta \circ \sigma_1 \circ z^{d_1/d}$$

for some linear function $\delta(z)$. Comparing critical points of the both sides of this equality we conclude that $\omega_r(z) = \gamma_r z$ for some $\gamma_r \in \mathbb{C}$.

Therefore, for $r \geq 2$ we have:

$$A_r(z) = \tilde{\sigma}_{r,1} \circ z^{(d_1/d)^r}, \quad B_r(z) = \tilde{\sigma}_{r,2} \circ z^{c_r} \tilde{R}_r(z^{(d_1/d)^r})$$

for some linear functions $\tilde{\sigma}_{r,1}, \tilde{\sigma}_{r,2}$ and a polynomial $\tilde{R}_r(z)$. In particular, setting $M_1 = \sigma_1^{-1}\{L_1\}$ and $M_r = \tilde{\sigma}_{r,1}^{-1}\{L_r\}$ for $r \geq 2$ we see that for any $r \geq 1$ the equality

$$(z^{(d_1/d)^r})^{-1}\{M_r\} = W\{K\} \tag{45}$$

holds.

Equality (45) implies that $W\{K\}$ together with a point u contains all the points of the form εu, where $\varepsilon^{(d_1/d)^r} = 1$ for some $r \geq 0$ and therefore all the circle $x^2 + y^2 = |u|$. It follows that $W\{K\}$ is a union of such circles and hence by corollary 2 the function $\tilde{f}_2(z)$ actually has the form $\sigma_2 \circ z^{d_2/d}$ for some linear function $\sigma_2(z)$. Furthermore, equality

$$T = \tilde{f}_1(W\{K\}) = \tilde{f}_2(W\{K\}) \tag{46}$$

implies that $\sigma_2(z) = \sigma_1 \circ \gamma z$ for some $\gamma \in \mathbb{C}$ and hence (8) holds.

To finish the proof we only must consider the case when $\deg \tilde{f}_1(z) = 2$. Define $A_2(z), B_2(z)$ by formula (35). Since $\deg A_2(z) = 4 > 2$ equality (36) implies that L_2 is either a union of circles or a segment and, respectively, $W\{K\}$ is either a union of circles centered at the origin or a segment $[-1, 1]$. If $W\{K\}$ is a union of circles then by corollary 2 we have:

$$\tilde{f}_1(z) = \sigma_1 \circ z^{d_1/d}, \quad \tilde{f}_2(z) = \sigma_2 \circ z^{d_2/d}$$

and as above equality (46) implies that (8) holds.

On the other hand, if $W\{K\}$ is a segment $[-1, 1]$ then the equality

$$\tilde{f}_1^{-1}\{T\} = T_2^{-1}\{[-1, 1]\}$$

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holds and applying corollary 1 we conclude that there exists a linear function σ_1 such that

$$T = \sigma_1\{[-1,1]\}, \quad \tilde{f}_1(z) = \sigma_1 \circ T_2.$$

Hence, T is a segment. Similarly,

$$\tilde{f}_2^{-1}\{T\} = T_{d_2/d}^{-1}\{[-1,1]\}$$

and

$$T = \sigma_2\{[-1,1]\}, \quad \tilde{f}_2(z) = \sigma_2 \circ T_{d_2/d}.$$

It follows now from $\sigma_1\{[-1,1]\} = \sigma_2\{[-1,1]\}$ that $\sigma_1^{-1} \circ \sigma_2 = \pm z$ and hence (9) holds.

6 Solutions of $f_1^{-1}\{T\} = f_2^{-1}\{T\} = T$

Proof of theorem 3. First of all consider the case when T is finite. In this case inequality (30) implies that T is a point, $T = t \in \mathbb{C}$. Let a_1 be the leading coefficient of $f_1(z)$. Set $\sigma(z) = \alpha(z - t)$, where $\alpha^{d_1 - 1} = a_1$. Then the polynomial $f(z) = \sigma \circ f_1 \circ \sigma^{-1}$ has the leading coefficient 1 and satisfies $f^{-1}\{0\} = 0$. It follows that $\sigma \circ f_1 \circ \sigma^{-1} = z^{d_1}$. Similarly, $\sigma \circ f_2 \circ \sigma^{-1} = \gamma z^{d_2}$ for some $\gamma \in \mathbb{C}$.

Assume now that T is infinite. Consider from the beginning the case when the set T is either a union of circles with the common center or a segment. Suppose first that T is a union of circles with the common center c. Without loss of generality we can assume that $c = 0$. Then it follows from corollary 2 that

$$f_1(z) = \gamma_1 z^{d_1}, \quad f_2(z) = \gamma_2 z^{d_2} \tag{47}$$

for some $\gamma_1, \gamma_2 \in \mathbb{C}$. Therefore, for $\sigma(z) = \alpha z$, where α satisfies $\alpha^{d_1 - 1} = \gamma_1$, the equalities (11) hold with $\gamma = \gamma_2 \alpha^{-d_2 + 1}$. Furthermore, if $r = \max_{z \in T} |z|$ then (47) and (10) imply that $\gamma_1 r^{d_1 - 1} = \gamma_2 r^{d_2 - 1}$. Therefore, since $r > 0$, the equality

$$\gamma_1^{d_2 - 1} = \gamma_2^{d_1 - 1} \tag{48}$$

holds and hence

$$|\gamma| = |\gamma_2||\alpha^{-d_2 + 1}| = |\gamma_2||\gamma_1^{-\frac{d_2 - 1}{d_1 - 1}}| = 1. \tag{49}$$

Similarly, if T is a segment then setting

$$p_1(z) = \sigma \circ T_{d_1} \circ \sigma^{-1}, \quad p_2(z) = \sigma \circ T_{d_2} \circ \sigma^{-1},$$

where $\sigma(z)$ is a linear function such that $T = \sigma\{[-1,1]\}$ we see that

$$p_1^{-1}\{T\} = p_2^{-1}\{T\} = T.$$

By the corollary 1 this implies that that

$$f_1(z) = \delta_1 \circ p_1(z), \quad f_2(z) = \delta_2 \circ p_2(z),$$

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for some linear function $\delta_1(z)$, $\delta_2(z)$ such that $\delta_1\{T\} = \delta_2\{T\} = T$. Since any linear function which transforms T to itself has the form

$$\delta(z) = \sigma \circ \pm z \circ \sigma^{-1}$$

it follows that (12) holds.

Consider now the case when T is distinct from a union of circles or a segment. Let $p(z)$, $s = \deg p(z)$, be a non-linear polynomial of the minimal degree satisfying

$$f^{-1}\{T\} = \{T\}. \tag{50}$$

Show that then for any polynomial $q(z)$, $t = \deg q(z)$, satisfying (50) the equality $t = s^k$ holds for some $k \geq 1$. Indeed, suppose that $s^k < t < s^{k+1}$ for some $k \geq 1$. Since

$$\left(p^{\circ k}(z)\right)^{-1}\{T\} = q^{-1}\{T\} = T$$

it follows from theorem 2 that there exists a polynomial $r(z)$ such that

$$q(z) = r(p(z)), \quad r^{-1}\{T\} = T.$$

Since $1 < \deg r(z) < s$ this contradicts to the assumption about $p(z)$.

Therefore, $\deg f_1(z) = s^{k_1}$, $\deg f_2(z) = s^{k_2}$ for some $k_1, k_2 \geq 1$. Since

$$\left(p^{\circ k_1}(z)\right)^{-1}\{T\} = f_1^{-1}\{T\}, \quad \left(p^{\circ k_2}(z)\right)^{-1}\{T\} = f_2^{-1}\{T\}$$

it follows now from corollary 1 that equalities (13) hold. Furthermore, since T is distinct from a union of circles corollary 2 implies that Σ_T is finite.

Proof of theorem 4. Prove at first the equivalence of conditions 1 and 2. Clearly, it is enough to show that any of the conclusions $a), b), c)$ in the formulation of theorem 3 implies that polynomials $f_1(z)$, $f_2(z)$ have the same Julia sets. In the cases $a), b)$ this is obvious so suppose that the case $c)$ holds. Denote by J_{f_1} and J_{f_2} the Julia sets of the polynomials $f_1(z)$, $f_2(z)$ and by K_{f_1} and K_{f_2} their filled-in Julia sets. Since for any polynomial $f(z)$ the equality $J_f = \partial K_f$ holds in order to prove the equality $J_{f_1} = J_{f_2}$ it is enough to prove that $K_{f_1} = K_{f_2}$.

Without loss of generality we can assume that the center of the disk of the smallest radius containing T is zero. Then

$$f_1(z) = \eta_1 p^{\circ s_1}, \quad f_2(z) = \theta_2 p^{\circ s_2} \tag{51}$$

for some b-th roots of unity η_1, θ_1. Furthermore, applying formula (31) to the polynomial $p(z)$ and taking into account that $\mu(z) = \alpha z$ for some $\alpha \in \mathbb{C}$ we see that $p(z) = z^a R(z^b)$ for some polynomial $R(z)$. This implies that for any $j \geq 1$ we have:

$$f_1^{\circ j}(z) = \eta_j p^{\circ s_1 j}, \quad f_2^{\circ j}(z) = \theta_j p^{\circ s_2 j}$$

for some b-th roots of unity η_j, θ_j. Therefore, the equalities

$$|f_1^{\circ j}(z)| = |p^{\circ s_1 j}|, \quad |f_2^{\circ j}(z)| = |p^{\circ s_2 j}|$$

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hold. This follows that $K_{f_1} = K_{f_2} = K_p$ and hence $J_{f_1} = J_{f_2} = J_p$.

Prove now the equivalence of conditions 2 and 3. Suppose first that 2 holds and set $J = J_{f_1} = J_{f_2}$. Then we have:

$$(f_1 \circ f_2)^{-1}\{J\} = (f_2 \circ f_1)^{-1}\{J\}.$$

It follows now from corollary 1 that (14) holds with $\mu \in \Sigma_J = J_{f_1} \cap J_{f_2}$.

Furthermore, if (14) holds and $\mu(z) = z$ in other words if $f_1(z), f_2(z)$ commute then the equality $J = J_{f_1} = J_{f_2}$ was already established by Julia [12] (for any rational functions) and can be proved easily as follows ([1]). Suppose that $z \in K_{f_1}$. Since (14) implies that

$$f_1(f_2^{\circ k}(z)) = f_2^{\circ k}(f_1(z))$$

we conclude that $f_1\{K_{f_2}\} \subset \{K_{f_2}\}$ for any $k \geq 1$. Hence, $f_1^{\circ j}\{K_{f_2}\} \subset \{K_{f_2}\}$ for any $j \geq 1$ and therefore $K_{f_2} \subset K_{f_1}$. By the symmetry also $K_{f_1} \subset K_{f_2}$ and hence $K_{f_1} = K_{f_2}$, $J_{f_1} = J_{f_2}$.

Consider now the case when $\mu(z) \neq z$. If $\mu^{\circ j}(z) \neq z$ for any j then both Σ_{T_1} and Σ_{T_2} are infinite and by corollary 2 taking into account that $\Sigma_{T_1} \cap \Sigma_{T_2} \neq \varnothing$ we conclude that Σ_{T_1} and Σ_{T_2} are unions of circles with the common center c. Furthermore, since (14) implies that for any linear function $\nu(z)$ the equality

$$\tilde{f}_1(\tilde{f}_2(z)) = \tilde{\mu}(\tilde{f}_2(\tilde{f}_1(z)))$$

holds with

$$\tilde{f}_1 = \nu \circ f_1 \circ \nu^{-1}, \quad \tilde{f}_2 = \nu \circ f_2 \circ \nu^{-1}, \quad \tilde{\mu} = \nu \circ \mu \circ \nu^{-1},$$

without loss of generality we can assume that $c = 0$. In this case the corollary 2 implies there exist $\gamma_1, \gamma_2 \in \mathbb{C}$ such that equalities (47) hold. Therefore, setting $\sigma(z) = \alpha z$, where $\alpha^{d_1-1} = \gamma_1$, we see that equalities (11) hold with $\gamma = \gamma_2\alpha^{-d_2+1}$. Moreover, equality (14) implies equality (48) and therefore equality (49). Hence, $J_{f_1} = J_{f_2}$.

Suppose now that $\mu(z)$ is a rotation of finite order d around a point c. As above we may assume that $c = 0$. Then $\mu(z) = \varepsilon_d z$ for some primitive d-th root of unity ε_d. Show that

$$f_1(z) = z^{a_1} R_1(z^d), \quad f_2(z) = z^{a_2} R_2(z^d) \tag{52}$$

for some polynomials $R_1(z), R_2(z)$ and integers a_1, a_2. Indeed, if both Σ_{T_1} and Σ_{T_2} are finite then, taking into account the equality $c = 0$, we conclude as above that there exist polynomials $\tilde{R}_1(z), \tilde{R}_2(z)$ and integers $\tilde{a}_1, \tilde{a}_2$ such that

$$f_1(z) = z^{\tilde{a}_1} \tilde{R}_1(z^{d_1}), \quad f_2(z) = z^{\tilde{a}_2} \tilde{R}_2(z^{d_2}),$$

where d_i is the order of Σ_{T_i}, $i = 1, 2$. Since $d|d_1, d|d_2$ this implies that (52) holds. On the other hand, if one of (or both) groups $\Sigma_{T_1}, \Sigma_{T_2}$ is infinite then it follows from (47) that (52) holds.

Following [3] define the polynomials

$$\tilde{f}_1(z) = z^{a_1} R_1^d(z), \quad \tilde{f}_2(z) = z^{a_2} R_2^d(z)$$

and show that they commute. Indeed, clearly

$$\tilde{f}_i \circ z^d = z^d \circ f_i, \quad i = 1, 2. \tag{53}$$

Therefore, we have:

$$\tilde{f}_1 \circ \tilde{f}_2 \circ z^d = \tilde{f}_1 \circ z^d \circ f_2 = z^d \circ f_1 \circ f_2 = z^d \circ \mu \circ f_2 \circ f_1 =$$

$$= z^d \circ f_2 \circ f_1 = \tilde{f}_2 \circ z^d \circ f_1 = \tilde{f}_2 \circ \tilde{f}_1 \circ z^d.$$

Hence, $\tilde{f}_1 \circ \tilde{f}_2 = \tilde{f}_2 \circ \tilde{f}_1$.

Furthermore, since $\tilde{f}_1(z)$ and $\tilde{f}_2(z)$ commute we have $K_{\tilde{f}_1} = K_{\tilde{f}_2}$. On the other hand, (53) implies that

$$\tilde{f}_i^{\circ j} \circ z^d = z^d \circ f_i^{\circ j}, \quad i = 1, 2$$

for any $j \geq 1$. This follows that

$$K_{f_i} = (z^d)^{-1}\{\tilde{K}_{f_i}\}, \quad i = 1, 2.$$

Hence, $K_{f_1} = K_{f_2}$ and $J_{f_1} = J_{f_2}$.

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